

SPINORIAL DESCRIPTION OF SU(3)- AND G₂-MANIFOLDS

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ABSTRACT. We present a uniform description of SU(3)-structures in dimension 6 as well as G₂-structures in dimension 7 in terms of a characterising spinor and the spinorial field equations it satisfies. We apply the results to hypersurface theory to obtain new embedding theorems, and give a general recipe for building conical manifolds. The approach also enables one to subsume all variations of the notion of a Killing spinor.

1. Introduction

This paper is devoted to a systematic and uniform description of SU(3)-structures in dimension 6, as well as G₂-structures in dimension 7, using a spinorial formalism. Any SU(3)- or G₂-manifold can be understood as a Riemannian spin manifold of dimension 6 or 7, respectively, equipped with a real spinor field ϕ or $\bar{\phi}$ of length one. Let us denote by ∇ the Levi-Civita connection and its lift to the spinor bundle. We prove that an SU(3)-manifold admits a 1-form η and an endomorphism field S such that the spinor ϕ solves, for any vector field X ,

$$\nabla_X \phi = \eta(X)j(\phi) + S(X) \cdot \phi,$$

where j is the Spin(6)-invariant complex structure on the spin representation space $\Delta = \mathbb{R}^8$ realising the isomorphism $\text{Spin}(6) \cong \text{SU}(4)$. In a similar vein, there exists an endomorphism \bar{S} such that the spinor $\bar{\phi}$ of a G₂-manifold satisfies the even simpler equation

$$\nabla_X \bar{\phi} = \bar{S}(X) \cdot \bar{\phi}.$$

We identify the characterising entities η , S , and \bar{S} with certain components of the intrinsic torsion and use them to describe the basic classes of SU(3)- and G₂-manifolds by means of a spinorial field equation. For example, it is known that nearly Kähler manifolds correspond to $S = \mu \text{Id}$ and $\eta = 0$ [Gr90], and nearly parallel G₂-manifolds are those with $\bar{S} = \lambda \text{Id}$ [FKMS97], since the defining equation reduces then to the classical constraint for a Riemannian Killing spinor. If S or \bar{S} is symmetric (and, in dimension 6, additionally $\eta = 0$), this is the equation defining generalised Killing spinors, which are known to correspond to half-flat structures [CS02] and cocalibrated G₂-structures [CS06]. For all other classes, Theorems 3.13 and 4.8 provide new information concerning ϕ and $\bar{\phi}$. To mention but one example, we shall characterise in Theorem 3.7 Riemannian spin 6-manifolds admitting a harmonic spinor of constant length. Theorem 4.8 states the analogue fact for G₂-manifolds.

We begin by reviewing algebraic aspects of the dimensions 6 and 7—and explain why it is more convenient to use, in the former case, real spinors instead of complex spinors. In section 3 we carefully relate the various geometric quantities cropping up in special Hermitian geometry, with particular care regarding: the vanishing or (anti-)symmetry of S, η , the intrinsic torsion, induced differential forms and Nijenhuis tensor, Lee and Kähler forms, and the precise spinorial PDE for ϕ . We introduce a connection well suited to describe the geometry, and its relationship to the more familiar characteristic connection. The same programme is then carried out in section 4 for G₂-manifolds. The first major application of this set-up occupies section 5: our results can

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be used to study embeddings of $SU(3)$ -manifolds in G_2 -manifolds and describe different types of cones (section 6). The latter results complement the first and last author's work [AH13]. This leads to the inception of a more unified picture relating the host of special spinor fields occurring in different parts of the literature: Riemannian Killing spinors, generalised Killing spinors, quasi-Killing spinors, Killing spinors with torsion etc. What we show in section 7 is that all those turn out to be special instances of the characterising spinor field equations for ϕ and $\bar{\phi}$ that we started with, and although looking, in general, quite different, these equations can be drastically simplified in specific situations.

The pattern that emerges here clearly indicates that the spinorial approach is not merely the overhaul of an established theory. Our point is precisely that it should be used to describe efficiently these and other types of geometries, like $SU(2)$ - or $Spin(7)$ -manifolds, and that it provides more information than previously known. Additionally, the explicit formulas furnish a working toolkit for understanding many different concrete examples, and for further study.

2. Spin linear algebra

The real Clifford algebras in dimensions 6, 7 are isomorphic to $\text{End}(\mathbb{R}^8)$ and $\text{End}(\mathbb{R}^8) \oplus \text{End}(\mathbb{R}^8)$ respectively. The spin representations are real and 8-dimensional, so they coincide as vector spaces, and we denote this common space by $\Delta := \mathbb{R}^8$. By fixing an orthonormal basis e_1, \dots, e_7 of the Euclidean space \mathbb{R}^7 , one choice for the real representation of the Clifford algebra on Δ is

$$\begin{aligned} e_1 &= +E_{18} + E_{27} - E_{36} - E_{45}, & e_2 &= -E_{17} + E_{28} + E_{35} - E_{46}, \\ e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48}, \\ e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &= +E_{14} - E_{23} - E_{58} + E_{67}, \\ e_7 &= +E_{12} - E_{34} - E_{56} + E_{78}, \end{aligned}$$

where the matrices E_{ij} denote the standard basis elements of the Lie algebra $\mathfrak{so}(8)$, i.e. the endomorphisms mapping e_i to e_j , e_j to $-e_i$ and everything else to zero.

We begin by discussing the 6-dimensional case. Albeit real, the spin representation admits a $Spin(6)$ -invariant complex structure $j : \Delta \rightarrow \Delta$ defined by the formula

$$j := e_1 \cdot e_2 \cdot e_3 \cdot e_4 \cdot e_5 \cdot e_6.$$

Indeed, $j^2 = -1$ and j anti-commutes with the Clifford multiplication by vectors of \mathbb{R}^6 ; this reflects the fact that $Spin(6)$ is isomorphic to $SU(4)$. The complexification of Δ splits,

$$\Delta \otimes_{\mathbb{R}} \mathbb{C} = \Delta^+ \oplus \Delta^-,$$

a consequence of the fact that j is a real structure making (Δ, j) complex-(anti)-linearly isomorphic to either Δ^\pm , via $\phi \mapsto \phi \pm i \cdot j(\phi)$. Any real spinor $0 \neq \phi \in \Delta$, furthermore, decomposes Δ into three pieces,

$$(2.1) \quad \Delta = \mathbb{R}\phi \oplus \mathbb{R}j(\phi) \oplus \{X \cdot \phi : X \in \mathbb{R}^6\}.$$

In particular, j preserves the subspaces $\{X \cdot \phi : X \in \mathbb{R}^6\} \subset \Delta$, and the formula

$$J_\phi(X) \cdot \phi := j(X \cdot \phi)$$

defines an orthogonal complex structure J_ϕ on \mathbb{R}^6 that depends on ϕ . Moreover, the spinor determines a 3-form by means of

$$\psi_\phi(X, Y, Z) := -(X \cdot Y \cdot Z \cdot \phi, \phi)$$

where the brackets indicate the inner product on Δ . The pair (J_ϕ, ψ_ϕ) is an $SU(3)$ -structure on \mathbb{R}^6 , and any such arises in this fashion from some real spinor. In certain cases this is an established construction: a nearly Kähler structure may be recovered from the Riemannian Killing spinor

[Gr90], for instance. All this can be summarised in the known fact that SU(3)-structures on \mathbb{R}^6 correspond one-to-one with real spinors of length one (mod \mathbb{Z}_2),

$$\mathrm{SO}(6)/\mathrm{SU}(3) \cong \mathbb{P}(\Delta) = \mathbb{RP}^7.$$

Example 2.1. Consider the spinor $\phi = (0, 0, 0, 0, 0, 0, 1) \in \Delta = \mathbb{R}^8$. With the basis chosen on p. 2, then, J_ϕ and ψ_ϕ read

$$J_\phi e_1 = -e_2, \quad J_\phi e_3 = e_4, \quad J_\phi e_5 = e_6, \quad \psi_\phi = e_{135} - e_{146} + e_{236} + e_{245},$$

where $e_{135} = e_1 \wedge e_3 \wedge e_5$ &c. Throughout this article e_i indicate tangent vectors and one-forms indifferently.

Below we summarise formulas expressing the action of J_ϕ and ψ_ϕ , whose proof is an easy exercise in local coordinates and so omitted.

Lemma 2.2. *For any unit spinors ϕ, ϕ^* and any vector $X \in \mathbb{R}^6$*

$$\psi_\phi \cdot \phi = -4 \cdot \phi, \quad \psi_\phi \cdot j(\phi) = 4 \cdot j(\phi), \quad \psi_\phi \cdot \phi^* = 0 \text{ if } \phi^* \perp \phi, j(\phi),$$

$$(X \lrcorner \psi_\phi) \cdot \phi = 2X \cdot \phi, \quad J_\phi \phi = 3j(\phi), \quad J_\phi(j(\phi)) = -3\phi.$$

In dimension 7 the space Δ does not carry an invariant complex structure akin to j . However, we still have a decomposition. If we take a non-trivial real spinor $0 \neq \phi \in \Delta$, we may split

$$(2.2) \quad \Delta = \mathbb{R}\phi \oplus \{X \cdot \phi : X \in \mathbb{R}^7\},$$

and we can still define a 3-form

$$\Psi_\phi(X, Y, Z) := (X \cdot Y \cdot Z \cdot \phi, \phi).$$

It turns out that Ψ_ϕ is stable (its GL(7)-orbit is open), and its isotropy group inside GL(7, \mathbb{R}) is isomorphic to the exceptional Lie group $G_2 \subset \mathrm{SO}(7)$. Thus we recover the renowned fact that there is a one-to-one correspondence between positive stable 3-forms $\Psi \in \Lambda^3 \mathbb{R}^7$ of fixed length and real lines in Δ :

$$\mathrm{SO}(7)/G_2 \cong \mathbb{P}(\Delta) = \mathbb{RP}^7.$$

In analogy to Lemma 2.2, here are formulas to be used in the sequel.

Lemma 2.3. *Let Ψ_ϕ be a stable three-form on \mathbb{R}^7 inducing the spinor ϕ , and suppose ϕ^* is a unit spinor orthogonal to ϕ . Then*

$$\Psi_\phi \cdot \phi = 7\phi, \quad \Psi_\phi \cdot \phi^* = -\phi^*, \quad (X \lrcorner \Psi_\phi) \cdot \phi = -3X \cdot \phi.$$

Remark 2.4. The existence of the unit spinor ϕ on M^6 is a general fact. Any 8-dimensional real vector bundle over a 6-manifold admits a unit section, see e.g. [Hu75, Ch. 9, Thm. 1.2]. Consequently, an oriented Riemannian 6-manifold admits a spin structure if and only if it admits a reduction from Spin(6) \cong SU(4) to SU(3). The argument also applies to Spin(7)- and G₂-structures, and was practised extensively in [FKMS97, Prop. 3.2].

The power of the approach presented in this paper is already manifest at this stage. Consider a 7-dimensional Euclidean space \bar{U} equipped with a G₂-structure $\Psi \in \Lambda^3 \bar{U}$. The latter induces an SU(3)-structure on any codimension-one subspace U , which may be defined in two ways. One can restrict Ψ to U , so that the inner product $V \lrcorner \Psi$ with a normal vector V defines a complex structure on U . But it is much simpler to remark that both structures, on \bar{U} and U , correspond to the same choice of the real spinor $\phi \in \Delta$.

3. Special Hermitian geometry

The premises now in place, an $SU(3)$ -manifold will be a Riemannian spin manifold (M^6, g, ϕ) equipped with a global spinor ϕ of length one. We always denote its spin bundle by Σ and the corresponding Levi-Civita connection by ∇ . The induced $SU(3)$ -structure is determined by the 3-form ψ_ϕ , while the 2-form $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$ defines the underlying $U(3)$ -structure. From now onwards we will drop the symbol for the Clifford product, so $X \cdot \phi$ will simply read $X\phi$.

Definition 3.1. By decomposition (2.1) there exist a unique one-form $\eta \in T^*M^6$ and a unique section $S \in \text{End}(TM^6)$ such that

$$(3.1) \quad \nabla_X \phi = \eta(X)j(\phi) + S(X)\phi.$$

We call S the *intrinsic endomorphism* and η the *intrinsic 1-form* of the $SU(3)$ -manifold (M^6, g, ϕ) ; this terminology will be fully justified by Proposition 3.3.

Recall that the geometric features of M^6 are captured [S89] (see also [A06] and [Fr03]) by the intrinsic torsion Γ which, under

$$\Lambda^2 T^*M^6 \cong \mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{su}(3)^\perp,$$

becomes a one-form with values in $\mathfrak{su}(3)^\perp$. For instance, nearly Kähler manifolds are those almost Hermitian manifolds for which Γ , identified with $\nabla\omega \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp$, is skew: $\nabla_X \omega(X, Y) = 0, \forall X, Y$. The aim is to recover the various $SU(3)$ -classes (complex, symplectic, lcK...) essentially by reinterpreting the intrinsic torsion using S and η . Besides ψ_ϕ , we have a second, so-to-speak fundamental 3-form

$$\psi_\phi^J(X, Y, Z) := \psi_\phi(JX, JY, JZ) = -\psi_\phi(JX, Y, Z) = -(XYZ\phi, j(\phi)),$$

which gives the imaginary part of a J_ϕ -holomorphic complex 3-form (the real part being ψ_ϕ). As a first result, we prove that the intrinsic torsion can be expressed through S and ψ_ϕ^J , while η is related to $\nabla\psi_\phi^J$ —thus generalizing the well-known definition of nearly Kähler manifolds cited above.

Lemma 3.2. *The intrinsic endomorphism S and the intrinsic 1-form η are related to $\nabla\omega$ and $\nabla\psi_\phi^J$ through (X, Y, Z) any vector fields)*

$$(\nabla_X \omega)(Y, Z) = 2\psi_\phi^J(S(X), Y, Z) \quad \text{and} \quad 8\eta(X) = -(\nabla_X \psi_\phi^J)(\psi_\phi).$$

Proof. We immediately find $\eta = (\nabla\phi, j(\phi))$. Since j can be thought of as the volume form, it is parallel under ∇ and we conclude

$$\nabla_X(j(\phi)) = j\nabla_X \phi = jS(X)\phi + j\eta(X)j(\phi) = -S(X)j(\phi) - \eta(X)\phi.$$

With $\omega(X, Y) = -(X\phi, Yj(\phi))$ we get

$$\begin{aligned} -\nabla_X \omega(Y, Z) &= X(Y\phi, Zj(\phi)) - (\nabla_X Y\phi, Zj(\phi)) - (Y\phi, \nabla_X Zj(\phi)) \\ &= (Y\nabla_X \phi, Zj(\phi)) + (Y\phi, Z\nabla_X j(\phi)) = (YS(X)\phi, Zj(\phi)) - (Y\phi, ZS(X)j(\phi)) \\ &= -2\psi_\phi^J(S(X), Y, Z). \end{aligned}$$

Similarly, we compute

$$\begin{aligned} \nabla_X(\psi_\phi^J)(\psi_\phi) &= -X(\psi_\phi\phi, j(\phi)) + (\nabla_X \psi_\phi\phi, j(\phi)) \\ &= -(\psi_\phi S(X)\phi, j(\phi)) + (\psi_\phi\phi, S(X)j(\phi)) - \eta(X)(\psi_\phi j(\phi), j(\phi)) + \eta(X)(\psi_\phi\phi, \phi) \\ &= 2\eta(X)(\psi_\phi\phi, \phi) = -8\eta(X). \end{aligned}$$

This finishes the proof. □

To understand the role of the pair (S, η) better we shall employ the $SU(3)$ -connection

$$(3.2) \quad \nabla_X^n Y = \nabla_X Y - \Gamma(X)(Y),$$

given by the Levi-Civita connection ∇ minus the intrinsic torsion, see [S89, Fr03]. We shall always use only one symbol for covariant derivatives on the tangent bundle and their liftings to the spinor bundle Σ , whence for any spinor ϕ^*

$$\nabla_X^n \phi^* = \nabla_X \phi^* - \frac{1}{2} \Gamma(X) \phi^*.$$

Proposition 3.3. *The intrinsic torsion of the SU(3)-structure (M^6, g, ϕ) is given by*

$$\Gamma = S \lrcorner \psi_\phi - \frac{2}{3} \eta \otimes \omega$$

where $S \lrcorner \psi_\phi(X, Y, Z) := \psi_\phi(S(X), Y, Z)$.

Proof. The spinor ϕ is parallel for ∇^n , as $\text{Stab}(\phi) = \text{SU}(3)$, so $\nabla_X \phi = \frac{1}{2} \Gamma(X) \phi$. By Lemma 2.2 we know that $\omega \phi = -3j(\phi)$, hence

$$\nabla_X \phi = S(X) \phi + \eta(X) j(\phi) = \frac{1}{2} (S(X) \lrcorner \psi_\phi) \phi - \frac{1}{3} \eta(X) \omega \phi.$$

Since $(X \lrcorner \psi_\phi)(Y, J_\phi Z) = (X \lrcorner \psi_\phi)(J_\phi Y, Z)$ we see that $X \lrcorner \psi_\phi \in \mathfrak{su}(3)^\perp$, and as $\omega \in \mathfrak{su}(3)^\perp$ the 1-form $S \lrcorner \psi_\phi - \frac{2}{3} \eta \otimes \omega$ is the intrinsic torsion of the spin connection. \square

Notation 3.4. The original approach to the classification of U(3)-manifolds in [GH80] was by the covariant derivative of the Kähler form. In analogy to their result, one calls the seven ‘basic’ irreducible modules of an SU(3)-manifold the *Gray-Hervella classes*. Throughout this paper they will be indicated $\chi_1^+, \chi_1^-, \chi_2^+, \chi_2^-, \chi_3, \chi_4, \chi_5$; for simplicity we often will write $\chi_j, \chi_{\bar{j}}$ for χ_j^+, χ_j^- respectively, and shorten $\chi_1^+ \oplus \chi_2^- \oplus \chi_4$ to $\chi_{1\bar{2}4}$, &c. In [CS02] the Gray-Hervella classes of SU(3)-manifolds were derived in terms of the components of the intrinsic torsion, while their identification with the covariant derivatives of the Kähler form and of the complex volume form may be found in [Ca05].

The following result links the intrinsic endomorphism S and the intrinsic 1-form η (and thus the spinorial field equation (3.1)) directly to the Gray-Hervella classes χ_i .

Lemma 3.5. *The basic classes of an SU(3)-structure (M^6, g, ϕ) are determined as follows, where $\lambda, \mu \in \mathbb{R}$:*

class	description	dimension
χ_1	$S = \lambda J_\phi, \eta = 0$	1
$\chi_{\bar{1}}$	$S = \mu \text{Id}, \eta = 0$	1
χ_2	$S \in \mathfrak{su}(3), \eta = 0$	8
$\chi_{\bar{2}}$	$S \in \{A \in S_0^2 T^* M \mid A J_\phi = J_\phi A\}, \eta = 0$	8
χ_3	$S \in \{A \in S_0^2 T^* M \mid A J_\phi = -J_\phi A\}, \eta = 0$	12
χ_4	$S \in \{A \in \Lambda^2(\mathbb{R}^6) \mid A J_\phi = -J_\phi A\}, \eta = 0$	6
χ_5	$S = 0, \eta \neq 0$	6

In particular, the class is $\chi_{1\bar{2}3}$ if and only if S is symmetric and η vanishes, recovering a result of [CS06].

3.1. Spinorial characterisation. The description of SU(3)-structures in terms of ϕ is the main result of this section. To start with, we discuss geometric quantities that pertain the SU(3)-structure and how they correspond to ϕ . Denote by D the Riemannian Dirac operator.

Lemma 3.6. *The χ_4 component of the intrinsic torsion of an $SU(3)$ -manifold is determined by*

$$\delta\omega(X) = 2[(D\phi, Xj(\phi)) - \eta(X)],$$

and in particular $\delta\omega = 0$ is equivalent to $(D\phi, Xj(\phi)) = \eta(X)$. The Lee form is given by

$$\theta(X) = \delta\omega \circ J(X) = 2(D\phi, X\phi) - 2\eta \circ J(X).$$

Proof. We have

$$(\nabla_X \omega)(Y, Z) = (ZY \nabla_X \phi, j(\phi)) + (ZY \phi, \nabla_X j(\phi)) = -2(YZ \nabla_X \phi, j(\phi)) - 2g(Y, Z)\eta(X),$$

leading to

$$\begin{aligned} \delta\omega(X) &= -\sum_i (\nabla_{e_i} \omega)(e_i, X) = \sum_i (\nabla_{e_i} \omega)(X, e_i) \\ &= -2 \sum_i ((Xe_i \nabla_{e_i} \phi, j(\phi)) - g(X, e_i)\eta(e_i)) \\ &= -2(XD\phi, j(\phi)) - 2\eta(X) = 2(D\phi, Xj(\phi)) - 2\eta(X). \end{aligned} \quad \square$$

We consider the space of all possible types $T^*M^6 \otimes \phi^\perp \ni \nabla\phi$, where $\phi^\perp = \mathbb{R}j(\phi) \oplus \{X\phi \mid X \in TM^6\}$ is the orthogonal complement of ϕ . The Clifford multiplication restricts then to a map

$$m : T^*M^6 \otimes \phi^\perp \rightarrow \Sigma.$$

Let $\pi : \text{Spin}(6) \rightarrow \text{SO}(6)$ be the usual projection. For any $h \in \text{Spin}(6)$ we have

$$m(\pi(h)\eta \otimes h\phi^*) = h\eta h^{-1}h\phi^* = hm(\eta \otimes \phi^*)$$

and m is $\text{Spin}(6)$ -equivariant and thus $SU(3)$ -equivariant. Comparing the dimensions of the modules appearing in (2.1) and the ones of Lemma 3.5 we see that $\chi_{2\bar{2}3} \subset \text{Ker}(m)$, and using

$$D\phi = 6\lambda j(\phi) \text{ for } S = \lambda J_\phi \quad \text{and} \quad D\phi = -6\mu\phi \text{ for } S = \mu \text{Id}$$

we find correspondences

$$\chi_1 \rightarrow \mathbb{R}j(\phi) \quad \text{and} \quad \chi_{\bar{1}} \rightarrow \mathbb{R}\phi,$$

together with $(D\phi, j(\phi)) = 6\lambda$ and $(D\phi, \phi) = -6\mu$.

Let us look at χ_{45} closer: recall that $\{J_\phi e_i \phi, \phi, j(\phi)\}$, $i = 1, \dots, 6$ is a basis of Σ for some local orthonormal frame e_i , hence

$$D\phi = \sum_{i=1}^6 (D\phi, J_\phi e_i \phi) J_\phi e_i \phi + (D\phi, \phi)\phi + (D\phi, j(\phi))j(\phi).$$

With Lemma 3.6 we conclude that

$$D\phi = \sum_{i=1}^6 [\frac{1}{2}\delta\omega(e_i) + \eta(e_i)]e_i j(\phi) + 6\lambda j(\phi) - 6\mu\phi = (\frac{1}{2}\delta\omega + \eta)j(\phi) + 6\lambda j(\phi) - 6\mu\phi.$$

Therefore, as image of m , the component \mathbb{R}^6 of Σ is determined by $\delta\omega + 2\eta$. This line of thought immediately proves

Theorem 3.7. *A 6-dimensional Riemannian spin manifold (M, g) carries a unit spinor ϕ lying in the kernel of the Dirac operator*

$$D\phi = 0$$

if and only if it admits an $SU(3)$ -structure of class $\chi_{2\bar{2}345}$ with the restriction $\delta\omega = -2\eta$.

The ‘complementary’ torsion components χ_1 and $\chi_{\bar{1}}$ are determined by the scalars

$$\lambda = \frac{1}{6}(D\phi, j(\phi)) = -\frac{1}{6}\text{tr}(J_\phi S) \quad \text{and} \quad \mu = -\frac{1}{6}(D\phi, \phi) = \frac{1}{6}\text{tr}(S).$$

One cannot but notice that harmonic spinors can exist on manifolds whose class is the opposite to that of nearly Kähler manifolds. The consequences of this observation remain – at this stage – to be seen, and will be addressed in forthcoming work.

The linear combination $\chi_4 + 2\chi_5$ vanishing in the theorem also shows up (up to a choice of volume) in [Ca03] and plays a role in supersymmetric compactifications of heterotic string theory.

Example 3.8. Consider the Lie algebra $\mathfrak{g} = \text{span}\{e_1, \dots, e_6\}$ with structure equations

$$\vec{d} = (e_{34} + 2e_{35}, e_{45}, 0, 0, 0, e_{51} + e_{23})$$

in terms of the Chevalley-Eilenberg differential $d\beta(e_i, e_j) = \beta[e_j, e_i], \forall \beta \in \mathfrak{g}$. Since the structure constants are rational the corresponding 1-connected Lie group G contains a co-compact lattice Λ . We consider the spin structure on $M^6 = G/\Lambda$ determined by choosing $\phi = (1, 1, 0, 0, 0, 0, 0)$. This gives us

$$S = -\frac{1}{4} \begin{bmatrix} 0 & 0 \\ -2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \eta = -\frac{1}{2}e_1$$

and it is not hard to see that $D\phi = 0$. The structure is of class $\chi_{2\bar{2}345}$, and the presence of component nr. 5 is reflected in the non-vanishing η .

Notation 3.9. Recalling Lemma 3.5 we decompose the intrinsic endomorphism into

$$S = \lambda J_\phi + \mu \text{Id} + S_2 + S_{34}$$

where J_ϕ commutes with S_2 , anti-commutes with S_{34} , and both S_2 and $J_\phi S_2$ are trace-free.

The next results discusses the Nijenhuis tensor $N_J(X, Y) = 8\text{Re}[X^{1,0}, Y^{1,0}]^{0,1}$, whose vanishing tells that M is a complex manifold. The customary trick in a Riemannian setting is to view it as a three-tensor $N(X, Y, Z) = g(N_J(X, Y), Z)$ by contracting with the metric.

Lemma 3.10. *The $\chi_{1\bar{1}2\bar{2}}$ component is controlled by the Nijenhuis tensor*

$$N(X, Y, Z) = -2[\psi_\phi^J((J_\phi S + S J_\phi)X, Y, Z) - \psi_\phi^J((J_\phi S + S J_\phi)Y, X, Z)].$$

Therefore if the class is $\chi_{1\bar{1}345}$, the Nijenhuis tensor reads

$$N(X, Y, Z) = 8[\lambda \psi_\phi^J(X, Y, Z) - \mu \psi_\phi(X, Y, Z)].$$

Proof. We have $g((\nabla_X J_\phi)Y, Z) = -(\nabla_X \omega)(Y, Z)$, and from Lemma 3.2

$$\begin{aligned} N(X, Y, Z) &= -(\nabla_X \omega)(J_\phi Y, Z) + (\nabla_Y \omega)(J_\phi X, Z) - (\nabla_{J_\phi X} \omega)(Y, Z) + (\nabla_{J_\phi Y} \omega)(X, Z) \\ &= 2[-\psi_\phi^J(SX, J_\phi Y, Z) + \psi_\phi^J(SY, J_\phi X, Z) - \psi_\phi^J(SJ_\phi X, Y, Z) + \psi_\phi^J(SJ_\phi Y, X, Z)] \\ &= 2[-\psi_\phi^J((J_\phi S + S J_\phi)X, Y, Z) + \psi_\phi^J((J_\phi S + S J_\phi)Y, X, Z)]. \end{aligned}$$

Futhermore for $S = \lambda J_\phi + \mu \text{Id} + S_{34}$ we have

$$J_\phi S + S J_\phi = J_\phi(S_{34} + \lambda J_\phi + \mu \text{Id}) + (S_{34} + \lambda J_\phi + \mu \text{Id})J_\phi = -2\lambda \text{Id} + 2\mu J_\phi,$$

as claimed. \square

Eventually, $\chi_{1\bar{1}34}$ depends on $d\omega$ in the following way:

Lemma 3.11. *Retaining Notation 3.9 we have*

$$d\omega(X, Y, Z) = 6\lambda \psi_\phi(X, Y, Z) + 6\mu \psi_\phi^J(X, Y, Z) + 2 \overset{XYZ}{\mathfrak{S}} \psi_\phi^J(S_{34}(X), Y, Z).$$

Proof. We have $d\omega(X, Y, Z) = \overset{XYZ}{\mathfrak{S}} (\nabla_X \omega)(Y, Z)$, and the fact that $\overset{XYZ}{\mathfrak{S}} \psi_\phi^J(S_2(X), Y, Z)$ vanishes corresponds to $d\omega = 0$ in $\chi_{2\bar{2}5}$. \square

To attain additional equations in terms of ϕ , thus completing the picture, we need one last technicality.

Lemma 3.12. *The intrinsic tensors (S, η) of a Riemannian spin manifold (M^6, g, ϕ) satisfy the following properties:*

$$\begin{aligned} S, J_\phi \text{ commute} &\iff (J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi), \\ S, J_\phi \text{ anti-commute} &\iff (J_\phi Y \nabla_X \phi, \phi) = (Y \nabla_{J_\phi X} \phi, \phi), \\ S \text{ is symmetric} &\iff (X \nabla_Y \phi, \phi) = (Y \nabla_X \phi, \phi), \\ S \text{ is skew-symmetric} &\iff (X \nabla_Y \phi, \phi) = -(Y \nabla_X \phi, \phi). \end{aligned}$$

Proof. As $(J_\phi S(X)\phi, Y\phi) = (SJ_\phi(X)\phi, Y\phi)$ if and only if $(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi(X)} \phi, \phi)$, the first two equivalences are clear.

Since both $\phi, j(\phi)$ are orthogonal to $Y\phi$, for any $Y \in TM^6$, we obtain

$$g(S(X), Y) = (\nabla_X \phi, Y\phi) \quad \text{and} \quad g(X, S(Y)) = (\nabla_Y \phi, X\phi)$$

and hence the remaining formulas. \square

Theorem 3.13. *The classification of $SU(3)$ -structures in terms of the defining spinor ϕ is contained in Table 3.1, where*

$$\eta(X) := (\nabla_X \phi, j(\phi))$$

and $\lambda = \frac{1}{6}(D\phi, j(\phi))$, $\mu = -\frac{1}{6}(D\phi, \phi)$ (as of Theorem 3.7).

Proof. We first prove that λ and μ in χ_1 and $\chi_{\bar{1}}$ are constant. In χ_1 we have $S = \lambda J_\phi$ and thus $\nabla_X(\phi + j(\phi)) = -\lambda X(\phi + j(\phi))$. Since a nearly Kähler structure (type $\chi_{1\bar{1}5}$) is given by a Killing spinor [Gr90], the function λ must be constant. In the case $\chi_{\bar{1}}$ the spinors ϕ and $j(\phi)$ themselves are Killing spinors with Killing constants $\mu, -\mu$.

We combine the results of Lemma 3.12 as follows. By Lemma 3.5, a structure is of type χ_2 if S is skew-symmetric, it commutes with J_ϕ , and the trace of $J_\phi S$ and η vanish. The first statement of Lemma 3.12 gives us the condition for S and J_ϕ to commute, and the last states that skew-symmetry is equivalent to $(X \nabla_Y \phi, \phi) = -(Y \nabla_X \phi, \phi)$, under which condition the equation $(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi)$ is equivalent to

$$(Y \nabla_X \phi, j(\phi)) = (X \nabla_Y \phi, j(\phi)).$$

The other classes can be calculated similarly, making extensive use of Lemmas 3.6, 3.12. \square

It makes little sense to compute all possible combinations (in principle, 2^7), so we listed only those of some interest. Others can be inferred by arguments of the following sort. Suppose we want to show that class χ_{124} has $(X \nabla_Y \phi, \phi) = -(Y \nabla_X \phi, \phi)$ and $\eta = 0$ as defining equations. From Lemma 3.5 we know χ_{124} is governed by the skew-symmetry of S , and at the same time η controls χ_5 , whence the claim is straightforward. Another example: assume we want to show that

$$3\lambda\psi_\phi(X, Y, Z) + 3\mu\psi_\phi^J(X, Y, Z) + \overset{XYZ}{\mathfrak{S}}(YZ\nabla_X \phi, j(\phi)) + \overset{XYZ}{\mathfrak{S}}\eta(X)g(Y, Z) = 0$$

is an alternative description of $\chi_{1\bar{1}2\bar{2}5}$. From Lemma 3.11 we know that $d\omega = 6\lambda\psi_\phi + 6\mu\psi_\phi^J$ defines that class, so we conclude by using $d\omega(X, Y, Z) = \overset{XYZ}{\mathfrak{S}}(\nabla_X \omega)(Y, Z)$ and the first equality in the proof of Lemma 3.6.

Remarks 3.14.

(i) The proof above shows that the real Killing spinors of an $SU(3)$ -structure of class $\chi_{1\bar{1}}$ (with Killing constants $\pm|\lambda|$) necessarily have the form $\phi \pm j(\phi)$ in case χ_1 , and $\phi, j(\phi)$ in case $\chi_{\bar{1}}$. Now notice that a rotation of ϕ to $\phi \cos \alpha + j(\phi) \sin \alpha$, for some function α , affects the intrinsic tensors as follows:

$$S \rightsquigarrow S \cos(2\alpha) + J_\phi \circ S \sin(2\alpha), \quad \eta \rightsquigarrow \eta + d\alpha$$

The χ_5 component varies, and $\chi_i^\pm, i = 1, 2$ change, too [CS02].

Table 3.1. Correspondence of SU(3)-structures and spinorial field equations (see Theorem 3.13).

<i>class</i>	<i>spinorial equations</i>
χ_1	$\nabla_X \phi = \lambda X j(\phi)$ for $\lambda \in \mathbb{R}$
$\chi_{\bar{1}}$	$\nabla_X \phi = \mu X \phi$ for $\mu \in \mathbb{R}$
χ_2	$(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi), \quad (Y \nabla_X \phi, j(\phi)) = +(X \nabla_Y \phi, j(\phi)), \quad \lambda = \eta = 0$
$\chi_{\bar{2}}$	$(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi), \quad (Y \nabla_X \phi, j(\phi)) = -(X \nabla_Y \phi, j(\phi)), \quad \mu = \eta = 0$
χ_3	$(J_\phi Y \nabla_X \phi, \phi) = +(Y \nabla_{J_\phi X} \phi, \phi), \quad (Y \nabla_X \phi, j(\phi)) = +(X \nabla_Y \phi, j(\phi)), \quad \eta = 0$
χ_4	$(J_\phi Y \nabla_X \phi, \phi) = +(Y \nabla_{J_\phi X} \phi, \phi), \quad (Y \nabla_X \phi, j(\phi)) = -(X \nabla_Y \phi, j(\phi)), \quad \eta = 0$
χ_5	$\nabla_X \phi = (\nabla_X \phi, j(\phi)) j(\phi)$
$\chi_{1\bar{1}}$	$\nabla_X \phi = \lambda X j(\phi) + \mu X \phi$
$\chi_{2\bar{2}}$	$(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi), \quad \lambda = \mu = 0$ and $\eta = 0$
$\chi_{2\bar{2}5}$	$(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi)$ and $\lambda = \mu = 0$
$\chi_{1\bar{1}2\bar{2}}$	$(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi)$ and $\eta = 0$
$\chi_{1\bar{1}2\bar{2}5}$	$(J_\phi Y \nabla_X \phi, \phi) = -(Y \nabla_{J_\phi X} \phi, \phi)$
$\chi_{2\bar{2}3}$	$D\phi = 0$ and $\eta = 0$
$\chi_{1\bar{1}2\bar{2}3}$	$(D\phi, X\phi) = 0$ and $\eta = 0$
$\chi_{1\bar{1}2\bar{2}34}$	$(\nabla_X \phi, j(\phi)) = 0$
$\chi_{2\bar{2}35}$	$(D\phi, Xj(\phi)) = \eta(X)$ and $\lambda = \mu = 0$
$\chi_{1\bar{1}2\bar{2}35}$	$(D\phi, Xj(\phi)) = \eta(X)$
χ_{34}	$(J_\phi Y \nabla_X \phi, \phi) = (Y \nabla_{J_\phi X} \phi, \phi)$ and $\eta = 0$
χ_{345}	$(J_\phi Y \nabla_X \phi, \phi) = (Y \nabla_{J_\phi X} \phi, \phi)$
$\chi_{2\bar{2}345}$	$\lambda = \mu = 0$
$\chi_{1\bar{1}23}$	$(X \nabla_Y \phi, \phi) = (Y \nabla_X \phi, \phi)$ and $\eta = 0$

In class $\chi_{1\bar{1}23}$ we have the constraint $D\phi = f\phi$, so ϕ is an eigenspinor with eigenfunction f . (One can alter ϕ so to have it in $\chi_{1\bar{1}2\bar{2}3}$.) Therefore, if we are after a Killing spinor (class $\chi_{1\bar{1}}$), the eigenfunction f necessarily determines the fifth component $\eta = -d\alpha$. In Section 6 we will treat cases where $f = h$ is a constant map.

(ii) It is fairly evident (cf. [CS02]) that the effect of modifying $S \rightsquigarrow JS$ is to exchange χ_j^+ and χ_j^- , $j = 1, 2$, whilst the other components remain untouched. As such it corresponds to a $\pi/2$ -rotation in the fibres of the natural circle bundle $\mathbb{RP}^7 \rightarrow \mathbb{CP}^3$.

Example 3.15. The twistor spaces $M^6 = \mathbb{CP}^3$, $U(3)/U(1)^3$ of the self-dual Einstein manifolds S^4 and \mathbb{CP}^2 are very interesting from the spinorial point of view. As is well known, both carry

a one-parameter family of metrics g_t compatible with two almost complex structures Ω^K, Ω^{nK} , in such a way that in a suitable, but pretty standard normalisation $(M^6, g_{1/2}, \Omega^{nK})$ is nearly Kähler and (M^6, g_1, Ω^K) is Kähler [ES85]. The two almost complex structures differ by an orientation flip on the two-dimensional fibres. Here is a short and uniform description of both instances. We choose the spin representation used in [BFGK91, Sect. 5.4], whereby the Riemannian scalar curvature of g_t is

$$\text{Scal}_t = 2c(6 - t + 1/t)$$

where c is a constant (equal to 1 for \mathbb{CP}^3 and $c = 2$ for $U(3)/U(1)^3$, due to an irrelevant yet nasty factor of 2 in standard normalisations). Using an appropriate orthonormal frame the orthogonal almost complex structures read

$$\Omega^K = e_{12} - e_{34} - e_{56}, \quad \Omega^{nK} = e_{12} - e_{34} + e_{56}.$$

There exist two linearly independent and isotropy-invariant real spinors ϕ_ε in Δ ($\varepsilon = \pm 1$), which define global spinor fields on the two spaces. One can prove directly that the ϕ_ε induce the same J_ϕ , corresponding to Ω^{nK} , and also the 3-forms

$$\psi_\varepsilon := \psi_{\phi_\varepsilon} = \varepsilon(e_{135} + e_{146} - e_{236} + e_{245}) =: \varepsilon\Psi.$$

When $t = 1/2$, ϕ_ε are known to be Killing spinors. For a generic $t \neq 0$ let us define the symmetric endomorphisms $S_\varepsilon : TM^6 \rightarrow TM^6$

$$S_\varepsilon = \varepsilon\sqrt{c} \cdot \text{diag}\left(\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{1-t}{2\sqrt{t}}, \frac{1-t}{2\sqrt{t}}\right).$$

An explicit calculation shows that ϕ_ε solve

$$\nabla_X \phi_\varepsilon = S_\varepsilon(X) \phi_\varepsilon,$$

making them generalised Killing spinors. In particular, S_ε are the intrinsic endomorphisms and $\eta = 0$; observe that S_ε commute with Ω^{nK} due to their block structure. By Lemma 3.5 the $SU(3)$ -structure defined by ϕ_ε is therefore of class $\chi_{1\bar{2}}$ for $t \neq 1/2$, and reduces to class $\chi_{\bar{1}}$ when $t = 1/2$.

The spinors ϕ_ε are eigenspinors of the Riemannian Dirac operator D with eigenvalues $6\mu = \text{tr} S_\varepsilon = \varepsilon\sqrt{c}\frac{t+1}{\sqrt{t}}$; they coincide, as they should, with the limiting values of Friedrich's general estimate [Fr80] when $t = 1/2$, and Kirchberg's estimate for Kähler manifolds [Ki86] for $t = 1$.

A further routine calculation shows that

$$\nabla_{e_i} \Omega^{nK} = \begin{cases} -\sqrt{ct} J e_i \lrcorner \Psi & 1 \leq i \leq 4 \\ -\frac{\sqrt{c}(1-t)}{\sqrt{t}} J e_i \lrcorner \Psi & i = 5, 6. \end{cases}$$

Hence, we conclude that $\nabla_X \Omega^{nK} = -2JS_\varepsilon(X) \lrcorner \psi_\varepsilon$ holds, as it should by Lemma 3.2.

Let us finish with a comment on the Kähler structures ($t = 1$). Kirchberg's equality is attained in odd complex dimensions by a pair of so-called Kählerian Killing spinors, basically ϕ_1, ϕ_{-1} [Ki88]. These, however, do *not* induce Ω^K , rather the 'wrong' almost complex structure Ω^{nK} . This means two things: first, the Kähler structure cannot be recovered from the two Kählerian Killing spinors; secondly, it reflects the fact that the Killing spinors do not define a 'compatible' $SU(3)$ -structure. For the projective space this stems from our description of \mathbb{CP}^3 as $SO(5)/U(2)$, on which there is no invariant spinor inducing Ω^K . In the other case the reason is that every almost Hermitian structure on the flag manifold is $SU(3)$ -invariant [AGI98].

3.2. Adapted connections. Let (M^6, g, ϕ) be an SU(3)-manifold with Levi-Civita connection ∇ . As we are interested in non-integrable structures, $\nabla\phi \neq 0$, we look for a metric connection that preserves the SU(3)-structure. The *canonical connection* defined in (3.2) is one such instance.

The space of metric connections is isomorphic to the space of $(2, 1)$ -tensors $\mathcal{A}^g := TM^6 \otimes \Lambda^2(TM^6)$ by $\tilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$. Define the map

$$\Xi : TM^6 \oplus \text{End}(TM^6) \rightarrow \mathcal{A}^g, (\eta, S) \mapsto -S \lrcorner \psi_\phi + \frac{2}{3}\eta \otimes \omega$$

where S, η are the intrinsic tensors of the SU(3)-structure on M^6 . Then $\nabla_X^n Y := \nabla_X Y + \Xi(\eta, S)$ is a metric connection on M^6 , and we get

$$\begin{aligned} \nabla_X^n \phi &= \nabla_X \phi - \psi_\phi \frac{1}{2}(S(X), \cdot, \cdot) \cdot \phi + \frac{1}{3}\eta(X)\omega \cdot \phi \\ &= S(X) \cdot \phi + \eta(X)j(\phi) - S(X) \cdot \phi - \eta(X)j(\phi) = 0 \end{aligned}$$

by Lemma 2.2, showing that ∇^n is an SU(3)-connection. The space \mathcal{A}^g splits under the representation of $\text{SO}(n)$ (see [Ca25, p.51] and [AF04]) into

$$\mathcal{A}^g = TM^6 \oplus \Lambda^3(TM^6) \oplus \mathcal{T},$$

whose summands are referred to as *vectorial*, *skew-symmetric* and *cyclic traceless* connections. A computer algebra system calculates the map Ξ at one point and gives

Lemma 3.16. *The ‘pure’ classes of an SU(3)-manifold M^6 correspond to ∇^n in:*

class of M^6	$\chi_{1\bar{1}}$	$\chi_{2\bar{2}}$	χ_3	χ_4	χ_5
type of ∇^n	Λ^3	\mathcal{T}	$\Lambda^3 \oplus \mathcal{T}$	$TM^6 \oplus \Lambda^3 \oplus \mathcal{T}$	$TM^6 \oplus \Lambda^3 \oplus \mathcal{T}$

The projection to the skew-symmetric part of the torsion given in the previous lemma generates the so-called *characteristic connection* ∇^c . This is a metric connection that preserves the SU(3)-structure and additionally has the same geodesics as ∇ . If an SU(3)-manifold admits such a connection, we know from [FI02] that the $\chi_{2\bar{2}}$ part of the intrinsic torsion vanishes.

We are interested in finding out whether and when an SU(3)-manifold (M^6, g, ϕ) admits a characteristic connection, that is to say when

$$\nabla^c \psi_\phi = 0.$$

Any connection doing that must be the (unique) characteristic connection of the underlying U(3)-structure, so to begin with the SU(3)-class must necessarily be $\chi_{1\bar{1}345}$. What is more,

Lemma 3.17. *Given an SU(3)-manifold (M^6, g, ϕ) , a connection with skew torsion $\tilde{\nabla}$ is characteristic if and only if it preserves the spinor ϕ .*

Proof. Obvious, but just for the record: ∇^c is an SU(3)-connection, and $\text{SU}(3) = \text{Stab}(\phi)$ forces ϕ to be parallel. Conversely, if ϕ is $\tilde{\nabla}$ -parallel, the connection must preserve any tensor arising in terms of the spinor, like ω and ψ_ϕ , cf. Lemma 3.2. To conclude, just recall that the characteristic connection is unique ([FI02], [AFH13]). \square

To obtain the ultimate necessary & sufficient condition we need to impose an additional constraint on χ_4, χ_5 :

Theorem 3.18. *A Riemannian spin manifold (M^6, g, ϕ) admits a characteristic connection if and only if it is of class $\chi_{1\bar{1}345}$ and $4\eta = \delta\omega$.*

Proof. Let ∇^c be the U(3)-characteristic connection, T its torsion. We shall determine in which cases $\nabla^c \phi = \nabla^c j(\phi) = 0$. First of all

$$0 = (\nabla_X^c \omega)(Y, Z) = -2(\nabla_X^c \phi, ZYj(\phi)) - 2g(Y, Z)(\nabla_X^c \phi, j(\phi)).$$

Consequently $(\nabla_X^c \phi, ZYj(\phi)) = 0$ if $Y \perp Z$. But as $Y \perp Z$ vary, the spinors $YZj(\phi)$ span ϕ^\perp . In conclusion, ∇^c is characteristic for the $SU(3)$ -structure iff $(\nabla_X^c \phi, j(\phi)) = 0$. Now choose a local adapted basis e_1, \dots, e_6 with $J_\phi e_i = -e_{i+1}, i = 1, 3, 5$. Using the formula

$$\nabla_X^c \phi = \nabla_X \phi + \frac{1}{4}(X \lrcorner T)\phi$$

and $\omega(X, Y) = -(XY\phi, j(\phi))$ we arrive at $4\eta(X) = -(X \lrcorner T\phi, j(\phi)) = \omega(X \lrcorner T) = T(\omega, X) = -1/2 \sum T(e_i, J_\phi e_i, X)$, and eventually

$$4\eta(X) = -\frac{1}{2} \sum_{i=1}^6 T(e_i, J_\phi e_i, X) = -\sum_{i=1}^6 (\nabla_{e_i} \omega)(e_i, X) = \delta\omega(X)$$

because $0 = (\nabla_X^c \omega)(Y, Z) = (\nabla_X \omega)(Y, Z) - \frac{1}{2}(T(X, J_\phi Y, Z) + T(X, Y, J_\phi Z))$. \square

The next theorem gives an explicit formula for the torsion of ∇^c . It relies on the computation for the Nijenhuis tensor of Lemma 3.10.

Suppose M^6 is of class $\chi_{1\bar{1}345}$, and decompose the intrinsic endomorphism into

$$S = \lambda J_\phi + \mu \text{Id} + S_{34},$$

as explained in Notation 3.9.

Theorem 3.19. *Suppose (M^6, g, ϕ) is of class $\chi_{1\bar{1}345}$. Then the characteristic torsion of the induced $U(3)$ -structure reads*

$$T(X, Y, Z) = 2\lambda\psi_\phi^J(X, Y, Z) - 2\mu\psi_\phi(X, Y, Z) - 2 \overset{XYZ}{\mathfrak{S}} \psi_\phi(S_{34}(X), Y, Z).$$

If $\eta = \frac{1}{4}\delta\omega$, T is the characteristic torsion of the $SU(3)$ -structure as well.

Proof. From Lemma 3.11 we infer

$$d\omega \circ J_\phi(X, Y, Z) = 6\lambda\psi_\phi^J(X, Y, Z) - 6\mu\psi_\phi(X, Y, Z) + 2 \overset{XYZ}{\mathfrak{S}} \psi_\phi(S_{34}(X), Y, Z)$$

The formula $T = N - d\omega \circ J$ (see [FI02]) together with Lemma 3.10 allows to conclude. \square

Remark 3.20. For the class $\chi_{1\bar{1}}$, the torsion T^c of the characteristic connection is parallel (for nearly Kähler manifolds, compare [Kir77, AFS05]). For such G -manifolds, the 4-form $\sigma_T := \frac{1}{2} \sum_i (e_i \lrcorner T) \wedge (e_i \lrcorner T)$ encodes a lot of geometric information. It is indeed equal to $dT/2$, it measures the non-degeneracy of the torsion, and it appears in the Bianchi identity, the Nomizu construction, and the identity for T^2 in the Clifford algebra (see [AFF13] where all these aspects are addressed). For the class $\chi_{1\bar{1}}$, an easy computation shows

$$\sigma_T = \lambda d\psi_\phi^J - \mu d\psi_\phi = 12(\lambda^2 + \mu^2) * \omega,$$

thus confirming the statement that σ_T encodes much of the geometry: it is basically given by the Kähler form.

Example 3.21. Take the real 6-manifold $M = \text{SL}(2, \mathbb{C})$ viewed as the reductive space

$$\frac{\text{SL}(2, \mathbb{C}) \times \text{SU}(2)}{\text{SU}(2)} = G/H$$

with diagonal embedding. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G and H , and set $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, so that

$$\mathfrak{m} = \{(A, B) \in \mathfrak{g} \mid A - \bar{A}^t = 0, \text{tr} A = 0, B + \bar{B}^t = 0, \text{tr} B = 0\}.$$

The almost complex structure

$$J(A, B) = (iA, iB)$$

defines a $U(3)$ -structure of class χ_3 , see [AFS05]. The characteristic connection $\nabla^c = \nabla + \frac{1}{2}T$ preserves a spinor ϕ , so ∇^c is also characteristic for the induced $SU(3)$ -structure, which is of class χ_{35} . By Theorem 3.18 we have $\eta = 0$, so actually the $SU(3)$ -class is χ_3 . But then ϕ is harmonic.

The following result shows that this reflects a more general fact:

Corollary 3.22. *Whenever ∇^c exists,*

$$\phi \in \text{Ker } D \iff T\phi = 0 \iff \text{the SU(3)-class is } \chi_3.$$

Proof. By Lemma 3.17, ϕ is ∇^c -parallel; since the Riemannian Dirac operator and the Dirac operator D^c of ∇^c are related by $D^c = D + \frac{3}{4}T$, the first equivalence follows. The equivalence of the first and the last statement is a direct consequence of Theorems 3.7, 3.18. \square

Example 3.21 satisfies $T\phi = 0$, as shown in [AFS05], so the first condition should be employed if more convenient. This example also shows that there exist SU(3)-structures different from type $\chi_{1\bar{1}5}$ (namely, χ_3) whose torsion is parallel.

4. G₂ geometry

Let (M^7, g, ϕ) be a Riemannian manifold with a globally defined unit spinor ϕ , inducing a G₂-structure Ψ_ϕ and the cross product \times :

$$\Psi_\phi(X, Y, Z) := (XYZ\phi, \phi) =: g(X \times Y, Z).$$

We recall two standard properties (see [FG82] or [AH13]):

Lemma 4.1. *The cross product and the 3-form Ψ_ϕ satisfy the identities*

- (1) $(X \times Y)\phi = -XY\phi - g(X, Y)\phi$
- (2) $*\Psi_\phi(V, W, X, Y) = \Psi_\phi(V, W, X \times Y) - g(V, X)g(W, Y) + g(V, Y)g(W, X).$

Motivated by the fact that $\{X\phi \mid X \in TM^7\} = \phi^\perp$, cf. (2.2), we have

Definition 4.2. There exists an endomorphism S of TM^7 satisfying

$$(4.1) \quad \nabla_X \phi = S(X)\phi$$

for every tangent vector X on M^7 , called the *intrinsic endomorphism* of (M^7, g, ϕ) .

Lemma 4.3. *The intrinsic endomorphism S satisfies*

$$(\nabla_V \Psi_\phi)(X, Y, Z) = 2 * \Psi_\phi(S(V), X, Y, Z).$$

Proof. We calculate

$$\begin{aligned} (\nabla_V \Psi_\phi)(X, Y, Z) &= (XYZ\nabla_V \phi, \phi) + (XYZ\phi, \nabla_V \phi) \\ &= (XYZS(V)\phi, \phi) - (S(V)XYZ\phi, \phi) \\ &= 2(XYZS(V)\phi, \phi) - 2g(S(V), Z)g(X, Y) + 2g(S(V), Y)g(X, Z) \\ &\quad - 2g(S(V), X)g(Y, Z). \end{aligned}$$

With Lemma 4.1 we get

$$\begin{aligned} 2 * \Psi_\phi(S(V), X, Y, Z) &= \\ &= -2[(XY(Z \times S(V))\phi, \phi) - g(X, Z)g(S(V), Y) + g(X, S(V))g(Y, Z)] \\ &= 2(XYZS(V)\phi, \phi) - 2g(Z, S(V))g(X, Y) + 2g(X, Z)g(S(V), Y) - 2g(X, S(V))g(Y, Z). \end{aligned}$$

\square

Proposition 4.4. *The intrinsic torsion of the G₂-structure Ψ_ϕ is*

$$\Gamma = -\frac{2}{3}S \lrcorner \Psi_\phi$$

where $S \lrcorner \Psi_\phi(X, Y, Z) := \Psi_\phi(S(X), Y, Z)$.

Proof. Immediate from Lemma 2.3, for $\frac{1}{2}\Gamma(X)\phi = \nabla_X \phi = S(X)\phi = -\frac{1}{3}(S(X) \lrcorner \Psi_\phi)\phi$. \square

To classify G_2 -structures one looks at endomorphisms of \mathbb{R}^7

$$\text{End}(\mathbb{R}^7) = \mathbb{R} \oplus S_0^2 \mathbb{R}^7 \oplus \mathfrak{g}_2 \oplus \mathbb{R}^7,$$

where $S_0^2 \mathbb{R}^7$ denotes symmetric, traceless endomorphisms of \mathbb{R}^7 . The original approach to the classification of G_2 -structures by Fernández-Gray [FG82] was by the covariant derivative of the 3-form Ψ_ϕ . In [Fr03] it was explained how the intrinsic torsion of G_2 -manifolds can be identified with $\text{End}(\mathbb{R}^7)$, thus yielding an alternative approach to the Fernández-Gray classes. The following result links the intrinsic endomorphism (and thus the spinorial field equation (4.1)) directly to the Fernández-Gray classes.

Lemma 4.5. *G_2 -structures fall into four basic types:*

<i>class</i>	<i>description</i>	<i>dimension</i>
\mathcal{W}_1	$S = \lambda \text{Id}$	1
\mathcal{W}_2	$S \in \mathfrak{g}_2$	14
\mathcal{W}_3	$S \in S_0^2 \mathbb{R}^7$	27
\mathcal{W}_4	$S \in \{V \lrcorner \Psi_\phi \mid V \in \mathbb{R}^7\}$	7

In particular, S is symmetric if and only if $S \in \mathcal{W}_1 \oplus \mathcal{W}_3$ and skew iff it belongs in $\mathcal{W}_2 \oplus \mathcal{W}_4$.

4.1. Spin formulation. By identifying $TM^7 \cong \phi^\perp$ we obtain the isomorphism $T^*M^7 \otimes TM^7 \cong T^*M^7 \otimes \phi^\perp$, given explicitly by

$$\eta \otimes X \mapsto \eta \otimes X\phi.$$

This enables us to describe the tensor product directly, through ϕ .

As in the $SU(3)$ case we will shorten $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ to \mathcal{W}_{134} and so on. The restricted Clifford product $m : T^*M^7 \otimes \phi^\perp \rightarrow \Delta$ decomposes the space \mathcal{W}_{1234} , as prescribed by the next result.

Theorem 4.6. *Let (M^7, g, ϕ) be a Riemannian spin manifold with unit spinor ϕ . Then ϕ is harmonic*

$$D\phi = 0$$

if and only if the underlying G_2 -structure is of class \mathcal{W}_{23} .

Proof. First of all, the spin representation splits as $\Delta = \mathbb{R}\phi \oplus \phi^\perp = \mathcal{W}_{14}$, so we may write the intrinsic-torsion space as

$$TM^7 \otimes \phi^\perp = \Delta \oplus \mathcal{W}_{23}.$$

Yet the multiplication m is G_2 -equivariant, so $\text{Ker } m = \{\sum_{ij} a_{ij} e_i \otimes e_j \phi \mid (a_{ij}) \in S_0^2 \mathbb{R}^7\} = \mathcal{W}_{23}$, and the assertion follows from the definition of $D = m \circ \nabla$. \square

Lemma 4.7. *In terms of ϕ the module \mathcal{W}_{24} depends on*

$$\frac{1}{2} \delta \Psi_\phi(X, Y) = (X\phi, \nabla_Y \phi) - (Y\phi, \nabla_X \phi) + (D\phi, XY\phi) + g(X, Y)(D\phi, \phi).$$

Proof. To prove the claim we simply calculate, in some orthonormal basis e_1, \dots, e_7 ,

$$\begin{aligned} \delta \Psi_\phi(X, Y) &= -\sum (\nabla_{e_i} \Psi_\phi)(e_i, X, Y) = -\sum [(XY e_i \nabla_{e_i} \phi, \phi) + (XY e_i \phi, \nabla_{e_i} \phi)] \\ &= -(XY D\phi, \phi) - \sum [-2g(e_i, Y)(X\phi, \nabla_{e_i} \phi) + 2g(e_i, X)(Y\phi, \nabla_{e_i} \phi) \\ &\quad + (e_i XY \phi, \nabla_{e_i} \phi)] \\ &= 2(D\phi, XY\phi) + 2(X\phi, \nabla_Y \phi) - 2(Y\phi, \nabla_X \phi) + 2g(X, Y)(D\phi, \phi). \end{aligned}$$

\square

At this point the complete picture is at hand.

Table 4.1. Correspondence of G₂-structures and spinorial field equations (see Theorem 4.8).

<i>class</i>	<i>spinorial equation</i>
\mathcal{W}_1	$\nabla_X \phi = \lambda X \phi$
\mathcal{W}_2	$\nabla_{X \times Y} \phi = Y \nabla_X \phi - X \nabla_Y \phi + 2g(Y, S(X))\phi$
\mathcal{W}_3	$(X \nabla_Y \phi, \phi) = (Y \nabla_X \phi, \phi)$ and $\lambda = 0$
\mathcal{W}_4	$\nabla_X \phi = X V \phi + g(V, X)\phi$ for some $V \in TM^7$
\mathcal{W}_{12}	$\nabla_{X \times Y} \phi = Y \nabla_X \phi - X \nabla_Y \phi + g(Y, S(X))\phi - g(X, S(Y))\phi - \lambda(X \times Y)\phi$
\mathcal{W}_{13}	$(X \nabla_Y \phi, \phi) = (Y \nabla_X \phi, \phi)$
\mathcal{W}_{14}	$\exists V, W \in TM^7 : \nabla_X \phi = X V W \phi - (X V W \phi, \phi)\phi$
\mathcal{W}_{23}	$S\phi = 0$ and $\lambda = 0$, or $D\phi = 0$
\mathcal{W}_{24}	$(X \nabla_Y \phi, \phi) = -(Y \nabla_X \phi, \phi)$
\mathcal{W}_{34}	$3(X\phi, \nabla_Y \phi) - 3(Y\phi, \nabla_X \phi) = (S\phi, XY\phi)$ and $\lambda = 0$
\mathcal{W}_{123}	$(S\phi, X\phi) = 0$, or $D\phi = -7\lambda\phi$
\mathcal{W}_{124}	$(Y \nabla_X \phi, \phi) + (X \nabla_Y \phi, \phi) = -2\lambda g(X, Y)$
\mathcal{W}_{134}	$3(X\phi, \nabla_Y \phi) - 3(Y\phi, \nabla_X \phi) = (S\phi, XY\phi) - 7\lambda g(X, Y)$
\mathcal{W}_{234}	$\lambda = 0$

Theorem 4.8. *The basic classes of G₂-manifolds are described by the spinorial field equations for ϕ as in Table 4.1. Here, $\lambda := -\frac{1}{7}(D\phi, \phi) : M \rightarrow \mathbb{R}$ is a real function, \times denotes the cross product relative to Ψ_ϕ , and*

$$S\phi := \sum_{i,j} g(e_i, S(e_j))e_i e_j \phi.$$

Proof. The proof relies on standard properties, like the fact that S is symmetric if and only if $(X \nabla_Y \phi, \phi) = (Y \nabla_X \phi, \phi)$. It could be recovered by going through the original argument of [FG82], but we choose an alternative approach.

For \mathcal{W}_1 there is actually nothing to prove, for the given equation is nothing but the Killing spinor equation characterising this type of manifolds [FK90, BFGK91].

The endomorphism S lies in \mathcal{W}_2 if and only if $S(X \times Y) = S(X) \times Y + X \times S(Y)$. Then

$$\begin{aligned} \nabla_{X \times Y} \phi &= (-Y \times S(X) + X \times S(Y))\phi \\ &= Y S(X)\phi + g(Y, S(X))\phi - X S(Y)\phi - g(X, S(Y))\phi \\ &= Y \nabla_X \phi - X \nabla_Y \phi + 2g(S(X), Y)\phi. \end{aligned}$$

By taking the dot product with ϕ we re-obtain that S is skew-symmetric.

For \mathcal{W}_3 we use Lemma 4.7:

$$\frac{1}{2}\delta\Psi_\phi(X, Y) = (D\phi, XY\phi) + g(X, Y)(D\phi, \phi) + (X\phi, \nabla_Y \phi) - (Y\phi, \nabla_X \phi).$$

This fact together with $\text{tr } S = -(D\phi, \phi)$ allows to conclude.

Suppose $S \in \mathcal{W}_4$. The vector representation \mathbb{R}^7 is $\{V \times \cdot \mid V \in \mathbb{R}^7\}$, so if S is represented by V we have $\nabla_X \phi = (V \times X)\phi = -VX\phi - g(V, X)\phi = XV\phi + g(V, X)\phi$.

As for the remaining ‘mixed’ types, we shall only prove what is not obvious. For type \mathcal{W}_{12} , S is of the form $S = \lambda \text{Id} + S'$, where $S' \in \mathfrak{g}_2$. Thus,

$$\nabla_{X \times Y} \phi = \lambda(X \times Y)\phi + S'(X \times Y)\phi.$$

The first term may be rewritten using Lemma 4.1 (1), while we deal with the second basically as we did in the pure \mathcal{W}_2 case. A clever rearrangement of terms then yields the desired identity.

By [FG82] a structure is of type \mathcal{W}_{23} if $(D\phi, \phi) = 0$ and $0 = \frac{1}{2} \sum_{i,j} \delta \Psi_\phi(e_i, e_j) \Psi_\phi(e_i, e_j, X)$. This is equivalent to

$$\begin{aligned} 0 &= \sum_{i,j} [(D\phi, e_i e_j \phi) - 7g(e_i, e_j)\lambda + (e_i \phi, S(e_j)\phi) - (e_j \phi, S(e_i)\phi)] (e_i e_j X \phi, \phi) \\ &= -\sum_{i,j} (D\phi, e_i e_j \phi) (e_i e_j \phi, X \phi) + 2 \sum_{i,j} (e_i \phi, S(e_j)\phi) (e_i e_j X \phi, \phi). \end{aligned}$$

As $\{e_i e_j \phi \mid i, j = 1, \dots, 7\}$ spans Δ , we obtain $\sum_{i,j} (\phi^*, e_i e_j \phi) e_i e_j \phi = 6\phi_1 + (\phi^*, \phi)\phi$. Define

$$S\phi := \sum_{i,j} g(e_i, S(e_j)) e_i e_j \phi$$

and get $0 = -6(D\phi, X\phi) - 2(S\phi, X\phi)$. Therefore $3D\phi = -S\phi$ holds on ϕ^\perp . If $\lambda = 0$ we then have

$$(S\phi, \phi) = \sum g(e_i, S(e_j)) (e_i e_j \phi, \phi) = -\sum g(e_i, S(e_i)) = (D\phi, \phi) = 0.$$

A structure is of type \mathcal{W}_{34} if $(D\phi, \phi) = 0$ and

$$3\delta \Psi_\phi(X, Y) = \frac{1}{2} \sum_{i,j} \delta \Psi_\phi(e_i, e_j) \Psi_\phi(e_i, e_j, X \times Y).$$

Due to the calculation above, the right-hand side equals

$$\begin{aligned} -6(D\phi, (X \times Y)\phi) &- 2(S\phi, (X \times Y)\phi) \\ &= 6(D\phi, XY\phi) - 42g(X, Y)\lambda + 2(S\phi, XY\phi) + 2g(X, Y)(S\phi, \phi). \end{aligned}$$

As

$$3\delta \Psi_\phi(X, Y) = 6(D\phi, XY\phi) - 42g(X, Y)\lambda + 6(X\phi, \nabla_Y \phi) - 6(Y\phi, \nabla_X \phi),$$

the defining equation is equivalent to

$$3(X\phi, \nabla_Y \phi) - 3(Y\phi, \nabla_X \phi) = (S\phi, XY\phi) - 7g(X, Y)\lambda$$

and if $\lambda = 0$ we get $3(X\phi, \nabla_Y \phi) - 3(Y\phi, \nabla_X \phi) = (S\phi, XY\phi)$.

As for \mathcal{W}_{124} , note that S satisfies $(Y\nabla_X \phi, \phi) + (X\nabla_Y \phi, \phi) = -2g(X, Y)\lambda$ if it is skew. If symmetric, instead, it satisfies the equation iff $g(X, S(Y)) = g(X, Y)\lambda$, i. e. if $S = \lambda \text{Id}$. \square

Remark 4.9. The spinorial equation for \mathcal{W}_4 defines a connection with vectorial torsion [AF06]. This is a G_2 -connection, since ϕ is parallel by construction.

4.2. Adapted connections. Let (M^7, g, ϕ) be a 7-dimensional spin manifold. As usual we identify $(3, 0)$ - and $(2, 1)$ -tensors using g . The prescription

$$\nabla^n := \nabla + \frac{2}{3} S \lrcorner \Psi_\phi$$

defines a natural G_2 -connection, since $(X \lrcorner \Psi_\phi)\phi = -3X\phi$, $\Psi_\phi \phi = 7\phi$ and $\Psi_\phi \phi^* = -\phi^*$, $\forall \phi^* \perp \phi$. Abiding by Cartan’s formalism, the set of metric connections is isomorphic to

$$\underbrace{\mathbb{R}^7}_{\Lambda^1 \mathbb{R}^7} \oplus \underbrace{(\mathbb{R} \oplus \mathbb{R}^7 \oplus S_0^2 \mathbb{R}^7)}_{\Lambda^3 \mathbb{R}^7} \oplus \underbrace{(\mathfrak{g}_2 \oplus S_0^2 \mathbb{R}^7 \oplus \mathbb{R}^{64})}_{\mathcal{T}}$$

under G_2 , and this immediately yields the analogous statement to Lemma 3.16:

Lemma 4.10. *The ‘pure’ classes of a G_2 -manifold (M^7, g, ϕ) correspond to ∇^n in:*

class of M^7	\mathcal{W}_1	\mathcal{W}_2	\mathcal{W}_3	\mathcal{W}_4
type of ∇^n	Λ^3	\mathcal{T}	$\Lambda^3 \oplus \mathcal{T}$	$TM^7 \oplus \Lambda^3$

Connections of type \mathcal{T} are rarely considered, although calibrated G₂-structures (\mathcal{W}_2) have an adapted connection of this type [CI07].

Among G₂-connections there exists at most one connection ∇^c with skew-symmetric torsion T^c . Therefore we may write

$$S = \lambda \text{Id} + S_3 + S_4 \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$$

with $S_3 \in S_0^2 TM^7$ and $S_4 = V \lrcorner \Psi_\phi$ for some vector V .

Proposition 4.11. *Let (M^7, g, ϕ) be a G₂-manifold of type \mathcal{W}_{134} . The characteristic torsion reads*

$$T^c(X, Y, Z) = -\frac{1}{3} \frac{XYZ}{\mathfrak{S}} \Psi_\phi((2\lambda \text{Id} + 9S_3 + 3S_4)X, Y, Z).$$

Proof. Consider the projections

$$\begin{aligned} T^*M^7 \otimes \mathfrak{g}_2^\perp &\xrightarrow{\kappa} \Lambda^3(T^*M^7) \xrightarrow{\Theta} T^*M^7 \otimes \mathfrak{g}_2^\perp \\ \Psi_\phi(SX, Y, Z) &\xrightarrow{\kappa} \frac{1}{3} \frac{XYZ}{\mathfrak{S}} \Psi_\phi(SX, Y, Z), \quad T \xrightarrow{\Theta} \sum_i e_i \otimes (e_i \lrcorner T)_{\mathfrak{g}_2^\perp} \end{aligned}$$

A little computation shows that the composite $\Theta \circ \kappa$ is the identity map, with eigenvalues 1, 0, 2/9, 2/3 on the four summands \mathcal{W}_i . But from [FI02] we know that if $-2\Gamma = \Theta(T)$ for some 3-form T , then T is the characteristic torsion. \square

5. Hypersurface theory

Let $(\bar{M}^7, \bar{g}, \phi)$ be a G₂-manifold and M^6 a hypersurface with transverse unit direction V

$$(5.1) \quad T\bar{M}^7 = TM^6 \oplus \langle V \rangle.$$

By restriction the spinor bundle $\bar{\Sigma}$ of \bar{M}^7 gives a Spin(6)-bundle Σ over M^6 , and so the Clifford multiplication \cdot of M^6 reads

$$X \cdot \phi = VX\phi$$

in terms of the one on \bar{M}^7 (whose symbol we suppress, as usual). This implies, in particular, that any $\sigma \in \Lambda^{2k} M^6 \subset \Lambda^{2k} \bar{M}^7$ of even degree will satisfy $\sigma \cdot \phi = \sigma\phi$. This notation was used in [BGM05] to describe almost Killing spinors (see Section 7). Caution is needed because this is not the same as $X \cdot \phi = X\phi$ described in Section 2 for comparing Clifford multiplications.

The second fundamental form $g(W(X), Y)$ of the immersion (W is the Weingarten map) accounts for the difference between the two Riemannian structures, and in $\bar{\Sigma}$ we can compare

$$\bar{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} VW(X)\phi.$$

A global spinor ϕ on \bar{M}^7 (a G₂-structure) restricts to a spinor ϕ on M^6 (an SU(3)-structure). The next lemma explains how both the almost complex structure and the spin structure are, essentially, induced by ϕ and the unit normal V .

Lemma 5.1. *For any section $\phi^* \in \Sigma$ and any vector $X \in TM^6$*

- (1) $V\phi^* = j(\phi^*)$
- (2) $VX\phi = (J_\phi X)\phi$.

Proof. The volume form σ_7 satisfies $\sigma_7 \phi^* = -\phi^*$ for any $\phi^* \in \Sigma$. Therefore $Vj(X\phi) = \sigma_7(X\phi) = -X\phi$. \square

This lemma is, at the level of differential forms, prescribing the rule $V \lrcorner \Psi_\phi = -\omega$.

Proposition 5.2. *With respect to decomposition (5.1) the intrinsic G_2 -endomorphism of \bar{M}^7 has the form*

$$(5.2) \quad \bar{S} = \begin{bmatrix} J_\phi S - \frac{1}{2} J_\phi W & * \\ \eta & ** \end{bmatrix}$$

where (S, η) are the intrinsic tensors of M^6 , J_ϕ the almost complex structure, W the Weingarten map of the immersion.

Proof. This result was first proved in [CS06] using the Cartan-Kähler machinery. Our argument is much simpler: the definitions imply $\nabla_X \phi = VS(X)\phi + \eta(X)V\phi$, and invoking Lemma 5.1 we infer $\bar{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} VW(X)\phi = J_\phi S(X)\phi - \frac{1}{2} J_\phi W(X)\phi + \eta(X)V\phi$. \square

The starred terms in (5.2) should point to the half-obvious fact that the derivative $\nabla_V \phi$ cannot be reconstructed from S and η . As a matter of fact, later we will show that the bottom row of \bar{S} is controlled by the product $(\nabla \phi, V\phi)$, so that the entry $**$ vanishes when $\nabla_V \phi = 0$.

Now we are ready for the main results, which explain how to go from M^6 to \bar{M}^7 (Theorem 5.4) and backwards (Theorem 5.5). The run-up to those requires a preparatory definition.

Recall that the Weingarten endomorphism W is symmetric if the $SU(3)$ -structure is half-flat (Lemma 3.5). Motivated by this

Definition 5.3. We say that a hypersurface $M^6 \subset \bar{M}^7$ has

- (0) *type zero* if W is the trivial map (meaning $\bar{\nabla} = \nabla$),
- (I) *type one* if W is of class χ_1 ,
- (II) *type two* if W is of class χ_2 ,
- (III) *type three* if W is of class χ_3 .

Due to the freedom in choosing entries in (5.2), we will take the easiest option (probably also the most meaningful one, geometrically speaking) and consider only embeddings where $\nabla_V \phi = 0$.

Theorem 5.4. *Embed (M^6, g, ϕ) in some $(\bar{M}^7, \bar{g}, \phi)$ as in (5.1), and suppose the G_2 -structure is parallel in the normal direction: $\bar{\nabla}_V \phi = 0$.*

Then the classes \mathcal{W}_α of $(\bar{M}^7, \bar{g}, \phi)$ depend on the column position (the class of M^6) and the row position (the Weingarten type of M^6) as in the table

	χ_1	χ_1	χ_2	χ_2	χ_3	χ_4	χ_5
<i>0</i>	\mathcal{W}_{13}	\mathcal{W}_4	\mathcal{W}_3	\mathcal{W}_2	\mathcal{W}_3	\mathcal{W}_{24}	\mathcal{W}_{234}
<i>I</i>	\mathcal{W}_{134}	\mathcal{W}_4	\mathcal{W}_{34}	\mathcal{W}_{24}	\mathcal{W}_{34}	\mathcal{W}_{24}	\mathcal{W}_{234}
<i>II</i>	\mathcal{W}_{123}	\mathcal{W}_{24}	\mathcal{W}_{23}	\mathcal{W}_2	\mathcal{W}_{23}	\mathcal{W}_{24}	\mathcal{W}_{234}
<i>III</i>	\mathcal{W}_{13}	\mathcal{W}_{34}	\mathcal{W}_3	\mathcal{W}_{23}	\mathcal{W}_3	\mathcal{W}_{234}	\mathcal{W}_{234}

Proof. Let A be an endomorphism of \mathbb{R}^6 and θ a covector. Then $\bar{A} = \begin{bmatrix} J_\phi A & 0 \\ \theta & 0 \end{bmatrix}$ is of type \mathcal{W}_4 iff $\theta = 0$ and A is a multiple of the identity, since J_ϕ is given by $g(X, J_\phi Y) = \frac{1}{2} \Psi_\phi(V, X, Y)$.

With similar, easy arguments one shows that the type of $\bar{A} = \begin{bmatrix} J_\phi A & 0 \\ \theta & 0 \end{bmatrix}$ is determined by the class of the intrinsic tensors (A, θ) on M^6 in the following way:

$(A, \theta) \in$	χ_1	$\chi_{\bar{1}}$	χ_2	$\chi_{\bar{2}}$	χ_3	χ_4	χ_5
$(J_\phi A, \theta) \in$	$\chi_{\bar{1}}$	χ_1	$\chi_{\bar{2}}$	χ_2	χ_3	χ_4	χ_5
$\bar{A} \in$	\mathcal{W}_{13}	\mathcal{W}_4	\mathcal{W}_3	\mathcal{W}_2	\mathcal{W}_3	\mathcal{W}_{24}	\mathcal{W}_{234}

Now the theorem can be proved thus: consider for example (S, η) of class χ_3 on a hypersurface of type I. Then $\begin{bmatrix} J_\phi S & 0 \\ \eta & 0 \end{bmatrix}$ has class \mathcal{W}_3 , and since W is a multiple of the identity $\begin{bmatrix} J_\phi W & 0 \\ 0 & 0 \end{bmatrix}$ has class \mathcal{W}_4 . This immediately gives $\bar{S} = \begin{bmatrix} J_\phi S - \frac{1}{2} J_\phi W & 0 \\ \eta & 0 \end{bmatrix}$, so the class of the G₂-structure is \mathcal{W}_{34} . All other cases are analogous. \square

With that in place we can now do the opposite: start from the ambient space $(\bar{M}^7, \bar{g}, \phi)$ and infer the structure of its codimension-one submanifolds M^6 . By inverting formula (5.2) we immediately see

$$S = -J_\phi \bar{S}|_{TM^6} + \frac{1}{2} W, \quad \eta(X) = g(\bar{S}X, V)$$

for any $X \in TM^6$. The next, final result on hypersurfaces can be found, in a different form, in [Ca06, Sect. 4].

Theorem 5.5. *Let $(\bar{M}^7, \bar{g}, \phi)$ be a Riemannian spin manifold of class \mathcal{W}_α . Then a hypersurface M^6 with normal $V \in T\bar{M}^7$ carries an induced spin structure ϕ : its class is an entry in the matrix below that is determined by the column (Weingarten type) and row position (\mathcal{W}_α)*

	\mathcal{W}_1	\mathcal{W}_2	\mathcal{W}_3	\mathcal{W}_4
0	χ_1	$\chi_{\bar{1}\bar{2}45}$	χ_{1235}	$\chi_{\bar{1}45}$
I	$\chi_{1\bar{1}}$	$\chi_{\bar{1}\bar{2}45}$	$\chi_{1\bar{1}235}$	$\chi_{\bar{1}45}$
II	$\chi_{1\bar{2}}$	$\chi_{\bar{1}\bar{2}45}$	$\chi_{12\bar{2}35}$	$\chi_{\bar{1}\bar{2}45}$
III	χ_{13}	$\chi_{\bar{1}\bar{2}345}$	χ_{1235}	$\chi_{\bar{1}345}$

Proof. In order to proceed as in Theorem 5.4, we prove that the class of an endomorphism $\bar{A} = \begin{bmatrix} J_\phi A & * \\ \theta & * \end{bmatrix}$ on \mathbb{R}^7 determines the class of (A, θ) on \mathbb{R}^6 in the following way:

$\bar{A} \in$	\mathcal{W}_1	\mathcal{W}_2	\mathcal{W}_3	\mathcal{W}_4
$(A, \theta) \in$	χ_1	$\chi_{\bar{1}\bar{2}45}$	χ_{1235}	$\chi_{\bar{1}45}$

If $\bar{A} \in \mathcal{W}_1$ we have $\bar{A} = \lambda \text{Id}$ and hence $\theta = 0$ and $A = \lambda J_\phi$.

If $\bar{A} \in \mathcal{W}_2$ then $J_\phi A$ is skew-symmetric, and A has type $\chi_{\bar{1}\bar{2}4}$.

If \bar{A} is of type \mathcal{W}_3 it follows $\bar{S} = \begin{bmatrix} J_\phi A & \eta \\ \eta & -\text{tr}(J_\phi A) \end{bmatrix}$ for some symmetric $J_\phi A$. Therefore JA is of type $\chi_{\bar{1}\bar{2}3}$, implying the type χ_{123} for A .

Suppose $\bar{A} \in \mathcal{W}_4$, so there is a vector Z such that $g(X, \bar{A}Y) = \Psi_\phi(Z, X, Y)$, whence

$$(XYZ\phi, \phi) = (\bar{A}Y\phi, X\phi)$$

for every $X, Y \in \mathbb{R}^7$. Restrict this equation to $X, Y \in \mathbb{R}^6$ and put $Z = \lambda V + Z_1, Z_1 \in \mathbb{R}^6$. Then $J_\phi A = \lambda J_\phi + A_1$ with $(XYZ_1\phi, \phi) = (A_1Y\phi, X\phi)$. Since A_1 is skew we have

$$\begin{aligned} g(X, A_1 J_\phi Y) &= (Z_1 X J_\phi Y \phi, \phi) = (Z_1 X V Y \phi, \phi) = -(Z_1 Y V X \phi, \phi) \\ &= -(Z_1 Y J_\phi X \phi, \phi) = -g(Y, A_1 J_\phi X) = -g(X, J_\phi A_1 Y), \end{aligned}$$

so $A_1 J_\phi = -J_\phi A_1$ and A_1 has type χ_4 . Eventually, $J_\phi A \in \chi_{14}$. \square

The above table explains why we cannot have a \mathcal{W}_1 -manifold if the derivative of ϕ along V vanishes. Moreover, in case $\nabla_V \phi = 0$ the χ_5 component disappears everywhere, simplifying the matter a little. Theorems 5.4 and 5.5 amend a petty mistake in [CS02, Thm 3.1] that was due to a (too) special choice of local basis.

6. Spin cones

We wish to explain how one can construct G_2 -structures, of any desired class, on cones over an $SU(3)$ -manifold. The recipe, which is a generalisation of the material presented in [AH13], goes as follows.

As usual, start with (M^6, g, ϕ) with intrinsic torsion (S, η) . Choose a complex-valued function $h : I \rightarrow S^1 \subset \mathbb{C}$ defined on some real interval I . Setting

$$\phi_t := h(t)\phi := \operatorname{Re} h(t)\phi + \operatorname{Im} h(t)j(\phi)$$

gives a new family of $SU(3)$ -structures on M^6 depending on $t \in I$, and $j(\phi)_t = j(\phi_t) = h(t)j(\phi)$. The product of a complex number $a \in \mathbb{C}$ with an endomorphism $A \in \operatorname{End}(TM)$ is defined as $aA = (\operatorname{Re} a)A + (\operatorname{Im} a)J_\phi A$. Then $h(A(X)\phi^*) = (hA)(X)\phi^* = A(X)\bar{h}\phi^*$ for any spinor ϕ^* . The first observation is that the intrinsic torsion of (M^6, g, ϕ_t) is given by $(h^2 S, \eta)$ (cf. Remark 3.14(ii), with $f = h$ constant on M^6), because

$$\begin{aligned} \nabla_X \phi_t &= h \nabla_X \phi = h(S(X) \cdot \phi) + h\eta(X)j(\phi) \\ &= (hS)(X) \cdot (\bar{h}h\phi) + \eta(X)j(\phi)_t = (h^2 S)(X) \cdot \phi_t + \eta(X)j(\phi)_t. \end{aligned}$$

If we rescale the metric conformally by some positive function $f : I \rightarrow \mathbb{R}_+$, we may consider

$$M_t^6 := (M^6, f(t)^2 g, \phi_t).$$

Note that M^6 and M_t^6 have the same Levi-Civita connection and spin bundle Σ , but distinct Clifford multiplications \cdot, \cdot_t , albeit related by $X \cdot \phi^* = \frac{1}{f(t)} X \cdot_t \phi^*, \forall \phi^*$. As

$$\nabla_X \phi_t = h^2 S(X) \cdot \phi_t + \eta(X)j(\phi)_t = \frac{h^2}{f} S(X) \cdot_t \phi_t + \eta(X)j(\phi)_t,$$

the intrinsic torsion of M_t^6 gets rescaled as $(\frac{h^2}{f} S, \eta)$.

Definition 6.1. The metric cone

$$(\bar{M}^7, \bar{g}) = (M^6 \times I, f(t)^2 g + dt^2)$$

equipped with spin structure $\bar{\phi} := \phi_t$ will be referred to as the *spin cone* over M^6 . The article [AH13] considered a version of this construction where $f(t) = t$.

The Levi-Civita connection $\bar{\nabla}^t$ of the cone reads

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{f'(t)}{f(t)} \bar{g}(X, Y) \partial_t$$

for $X, Y \in TM^6$, whence the Weingarten map is $W = -\frac{f'}{f} \operatorname{Id}$. Furthermore,

$$\bar{\nabla}_{\partial_t} \bar{\phi} = \bar{\nabla}_{\partial_t} h\phi = h' \phi = -i h' j(\phi) = -i \frac{h'}{h} h V \phi = -i \frac{h'}{h} V \bar{\phi}.$$

To sum up, the intrinsic torsion of \bar{M}^7 is encoded in

$$\bar{S} = \begin{bmatrix} \frac{h^2}{f} J_\phi S + \frac{f'}{2f} J_\phi & 0 \\ \eta & -i \frac{h'}{h} \end{bmatrix}.$$

By decomposing $S = \lambda J_\phi + \mu \operatorname{Id} + R \in \chi_1 \oplus \chi_{\bar{1}} \oplus \chi_{2\bar{2}345}$, the upper-left term in the matrix \bar{S} can be written as

$$\frac{-\lambda \operatorname{Im} h^2 + \mu \operatorname{Re} h^2 + f'/2}{f} J_\phi - \frac{\lambda \operatorname{Re} h^2 + \mu \operatorname{Im} h^2}{f} \operatorname{Id} + \frac{\operatorname{Re} h^2}{f} J_\phi R - \frac{\operatorname{Im} h^2}{f} R.$$

Let us see what happens for specific choices of hypersurface structure.

Suppose we require \bar{M}^7 to be a nearly integrable G₂-manifold (class \mathcal{W}_1): since \bar{S} is then a multiple of the identity, we need h'/h to be constant, so $h(t) = \exp(i(ct + d))$, $c, d \in \mathbb{R}$. The easiest instance of this situation is the following:

The sine cone. Start with an SU(3)-manifold (M^6, g, ϕ) of type $\chi_{\bar{1}}$ with $S = -\frac{1}{2}\text{Id}$. The choice $h = e^{it/2}$ produces a cone

$$(M^6 \times (0, \pi), \sin(t)^2 g + dt^2, e^{it/2} \phi)$$

for which $\bar{S} = \frac{1}{2}\text{Id}$. This construction was introduced in [ADHL03], see also [FIVU08, S09].

Cones of pure class. To obtain other classes of G₂-manifolds we start this time by fixing the function $h = 1$, so that $\bar{\phi} = \phi$ and

$$\bar{S} = \begin{bmatrix} \frac{\mu + \frac{1}{2}f'}{f} J_\phi - \frac{\lambda}{f} \text{Id} + \frac{1}{f} J_\phi R & 0 \\ \eta & 0 \end{bmatrix},$$

and only now we prescribe the SU(3)-structure.

- a) Take M^6 to be $\chi_{\bar{1}\bar{2}}$, say $S = \mu \text{Id} + R$, and $\mu < 0$ constant: the cone

$$(M^6 \times \mathbb{R}_+, 4\mu^2 t^2 g + dt^2, \phi)$$

has $\bar{S} = \begin{bmatrix} -\frac{1}{2\mu t} J_\phi R & 0 \\ 0 & 0 \end{bmatrix}$, and so it carries a calibrated G₂-structure (class \mathcal{W}_2).

- b) On M^6 of type $\chi_{\bar{1}\bar{2}\bar{3}}$ with $\mu < 0$ constant, we can build the same cone as in a), but now the resulting G₂-structure will be balanced (class \mathcal{W}_3).

- c) Take a $\chi_{\bar{1}}$ -manifold ($S = \mu \text{Id}$). Since $\begin{bmatrix} k(t)J_\phi & 0 \\ 0 & 0 \end{bmatrix}$ is of type \mathcal{W}_4 irrespective of the map $k(t)$, the cone

$$(M^6 \times I, f(t)^2 g + dt^2, \phi)$$

is always \mathcal{W}_4 , since R and λ vanish. When $\mu < 0$, the special choice $f(t) = -2\mu t$ will additionally give $\bar{S} = 0$. This Ansatz was used in [Bä93] to manufacture a parallel G₂-structure (trivial class $\{0\}$) on the cone.

Other choices of SU(3)-class on M^6 and functions h, f will allow, along these lines, to construct any desired G₂-class on a suitable cone.

7. Killing spinors with torsion

Let $(\bar{M}^7, \bar{g}, \phi)$ be a G₂-manifold with characteristic connection $\bar{\nabla}^c$ and torsion \bar{T} , and suppose (M^6, g, ϕ) is a submanifold of type I or III such that $V \lrcorner \bar{T} = 0$, cf. (5.1). The latter equation warrants that \bar{T} restricts to a 3-form on M^6 ; observe that the condition is more restrictive than assuming $\bar{\nabla}_V \phi = 0$, which implies only $(V \lrcorner \bar{T})\phi = 0$.

We decompose the Weingarten map $W = \mu \text{Id} + W_3$ with $JW_3 = -W_3J$ and prove

Lemma 7.1. *The differential form*

$$L(X, Y, Z) := -\overset{XYZ}{\mathfrak{S}} \psi_\phi(W_3(X), Y, Z) - \mu \psi_\phi(X, Y, Z)$$

satisfies $(X \lrcorner L)\phi = -2W(X)\phi$.

Proof. In an arbitrary orthonormal basis e_1, \dots, e_6 the torsion is $-\sum_i (e_i \lrcorner T)_{\mathfrak{su}(3)^\perp} \otimes e_i = 2\Gamma$, where $(e_i \lrcorner T)_{\mathfrak{su}(3)^\perp}$ denotes the projection of $e_i \lrcorner T$ under $\mathfrak{so}(6) \rightarrow \mathfrak{su}(3)^\perp$. It is not hard to see

that the maps

$$T^*M^6 \otimes \mathfrak{su}(3)^\perp \xrightarrow{\kappa} \Lambda^3(T^*M^6) \xrightarrow{\Theta} T^*M^6 \otimes \mathfrak{su}(3)^\perp$$

$$S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \omega \xrightarrow{\kappa} \frac{1}{3} \mathfrak{S} (S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \omega), \quad T \xrightarrow{\Theta} \sum_i e_i \otimes (e_i \lrcorner T)_{\mathfrak{su}(3)^\perp}$$

satisfy $\Theta \circ \kappa|_{\chi_3} = \frac{1}{3}\text{Id}_{\chi_3}$ and $\Theta \circ \kappa|_{\chi_1} = \text{Id}_{\chi_1}$. But since $\text{SU}(3)$ is the stabiliser of ϕ , for any $R \in \Lambda^3 T^*M^6$ we have $R(X)\phi = \Theta(R)(X)\phi$, so

$$(X \lrcorner L)\phi = -(\psi_\phi \lrcorner W)\phi = -2W(X)\phi,$$

proving the lemma. \square

For $X \in TM^6$ we have

$$0 = \bar{\nabla}_X^c \phi = \bar{\nabla}_X \phi + \frac{1}{4}(X \lrcorner \bar{T})\phi = \nabla_X \phi + \frac{1}{4}(X \lrcorner \bar{T})\phi - \frac{1}{2}W(X)\phi.$$

So if we define

$$T := \bar{T}|_{M^6} + L,$$

then $\nabla^c := \nabla + T$ is characteristic for (M^6, g, ϕ) . This means that if \bar{M}^7 and M^6 admit characteristic connections, their difference must be L .

Definition 7.2. Consider the one-parameter family of metric connections

$$\nabla^s := \nabla + 2sT$$

passing through ∇^c at $s = 1/4$ and ∇ at the origin. A spinor ϕ^* is called a *generalised Killing spinor with torsion* (gKST) if

$$\nabla_X^s \phi^* = A(X)\phi^*$$

for some symmetric $A : TM^6 \rightarrow TM^6$. This notion captures many old acquaintances: taking $s = 0$ will produce generalised Killing spinors (without torsion) [BGM05, CS07], and quasi-Killing spinors on Sasaki manifolds for special A [FK01]. Killing spinors with torsion correspond to $A = \text{Id}$, $s \neq 0$ [ABBK13], while ordinary Killing spinors arise of course from $s = 0$ and $A = \text{Id}$ [Fr80, BFGK91]. Our treatment intends to subsume all these notions into one and shed light on the mutual relationships.

Example 7.3. In view of Lemma 4.5, any cocalibrated G_2 -manifold (class \mathcal{W}_{13}) is defined by a gKS. For example, the standard G_2 -structure of a 7-dimensional 3-Sasaki manifold is cocalibrated, and indeed the *canonical spinor* is generalised Killing [AF10].

Suppose that ϕ^* , restricted to M^6 , is a gKST. Then at any point of M^6

$$\begin{aligned} \bar{\nabla}_X^s \phi^* &= \bar{\nabla}_X \phi^* + s(X \lrcorner \bar{T})\phi^* = \nabla_X \phi^* + s(X \lrcorner \bar{T})\phi^* - \frac{1}{2}VW(X)\phi^* \\ &= \nabla_X^s \phi^* + s(X \lrcorner (\bar{T} - T))\phi^* - \frac{1}{2}VW(X)\phi^* \\ &= V(A - \frac{1}{2}W)(X)\phi^* - s(X \lrcorner L)\phi^*. \end{aligned}$$

Picking $A = \frac{1}{2}W$ annihilates the first term, so we are left with $\bar{\nabla}_X^s \phi^* = -s(X \lrcorner L)\phi^*$. Conversely, any $\bar{\nabla}^s$ -parallel spinor on \bar{M}^7 satisfies

$$0 = \bar{\nabla}_X^s \phi^* = \nabla_X^s \phi^* + s(X \lrcorner (\bar{T} - T))\phi^* - \frac{1}{2}W(X)\phi^*.$$

To sum up,

Theorem 7.4. Let $(\bar{M}^7, \bar{g}, \phi)$ be a G_2 -manifold with characteristic connection $\bar{\nabla}^c$ and torsion \bar{T} . Take a hypersurface $M^6 \subset \bar{M}^7$ of type one or three such that $V \lrcorner \bar{T} = 0$. Then

- (1) $(M^6, g = \bar{g}|_{TM^6}, \phi)$ is an $\text{SU}(3)$ -manifold with characteristic connection $\nabla + \bar{T} + L$;
- (2) any solution ϕ^* on \bar{M}^7 to the gKST equation $\nabla_X^s \phi^* = \frac{1}{2}W(X)\phi^*$ on M^6 must satisfy

$$\bar{\nabla}_X^s \phi^* = -s(X \lrcorner L)\phi^*;$$

(3) Vice versa, if ϕ^* is $\bar{\nabla}^s$ -parallel on \bar{M}^7 , it solves

$$\nabla_X^s \phi^* = -sX \lrcorner (\bar{T} - T) \phi^* + \frac{1}{2}W(X) \phi^*.$$

Example 7.5. Given (M^6, g) we build the twisted cone

$$(\bar{M}^7 := M^6 \times \mathbb{R}, \bar{g} := a^2 t^2 g + dt^2)$$

for some $a > 0$. From the submanifold $M^6 \cong M^6 \times \{\frac{1}{a}\} \subset \bar{M}^7$ we can only infer the Clifford multiplication of \bar{M}^7 at points of $M^6 \times \{\frac{1}{a}\}$. Therefore we consider, as in Section 6, the hypersurface $M_t^6 := (M^6, a^2 t^2 g) \cong M^6 \times \{\frac{t}{a}\} \subset \bar{M}^7$. At any point in M_t^6 the spinor bundles of M_t^6 and \bar{M}^7 are the same and can be identified with the spinor bundle of M^6 . Hence $X \phi^* = \frac{1}{at} \partial_t X \phi^*$. Since the metric of M_t^6 is just a rescaling of that of M^6 , the Levi-Civita connections ∇ coincide. For the Riemannian connection $\bar{\nabla}$ on \bar{M}^7 we have

$$\bar{\nabla}_X \phi^* = \nabla_X \phi^* + \frac{1}{2t} \partial_t X \phi^* = \nabla_X \phi^* + \frac{a}{2} X \phi^*,$$

as $W(X) = -\frac{1}{t}X$. Therefore the submanifolds M_t^6 are of type I, and one can determine the possible structures using Theorems 5.4, 5.5. Any 2-form σ on M^6 is a 2-form on \bar{M}^7 with $\partial_t \lrcorner \sigma = 0$, and in addition

$$\sigma \cdot \phi^* = a^2 t^2 \sigma \phi^*$$

for any spinor ϕ^* .

Let ϕ be an SU(3)-structure on M^6 and consider the G₂-structure on \bar{M}^7 given by ϕ . Then $\partial_t \lrcorner \Psi_\phi = -a^2 t^2 \omega$. If M^6 has characteristic connection ∇^c with torsion T ,

$$\begin{aligned} 0 &= \nabla_X^c \phi = \nabla_X \phi + \frac{1}{4}(X \lrcorner T) \phi = \bar{\nabla}_X \phi - \frac{a}{2} X \phi + \frac{1}{4}(X \lrcorner T) \phi \\ &= \bar{\nabla}_X \phi - \frac{a}{4}(X \lrcorner \psi_\phi) \phi + \frac{1}{4}(X \lrcorner T) \phi = \bar{\nabla}_X \phi + \frac{1}{4}(X \lrcorner (T - a\psi_\phi)) \phi \\ &= \bar{\nabla}_X \phi + \frac{1}{4}(X \lrcorner a^2 t^2 (T - a\psi_\phi)) \phi, \end{aligned}$$

showing that $\bar{T} = a^2 t^2 (T - a\psi_\phi)$ is the characteristic torsion of \bar{M}^7 .

Given a $\bar{\nabla}^s$ -parallel spinor ϕ^*

$$\begin{aligned} 0 &= \bar{\nabla}_X \phi^* + s(X \lrcorner \bar{T}) \phi^* = \nabla_X \phi^* + \frac{a}{2} X \phi^* + \frac{s}{a^2 t^2} (X \lrcorner \bar{T}) \phi^* \\ &= \nabla_X \phi^* + \frac{a}{2} X \phi^* + s(X \lrcorner (T - a\psi_\phi)) \phi^* = \nabla_X^s \phi^* + \frac{a}{2} X \phi^* - as(X \lrcorner \psi_\phi) \phi^*, \end{aligned}$$

from which

$$\nabla_X^s \phi^* - as(X \lrcorner \psi_\phi) \phi^* = -\frac{a}{2} X \phi^*.$$

Consider the differential form on \bar{M}^7

$$\bar{\psi}_\phi(X, Y, Z) := a^3 t^3 \psi^-(X, Y, Z) \text{ for } X, Y, Z \in TM^6 \text{ and } \partial_t \lrcorner \bar{\psi}_\phi = 0$$

For a Killing spinor solving $\nabla_X^s \phi^* = -\frac{a}{2} X \phi^*$ we then have

$$\begin{aligned} 0 &= \nabla_X^s \phi^* + \frac{a}{2} X \phi^* = \nabla_X \phi^* + \frac{a}{2} X \phi^* + s(X \lrcorner T) \phi^* \\ &= \bar{\nabla}_X \phi^* + sa^2 t^2 (X \lrcorner T) \phi^* = \bar{\nabla}_X \phi^* + sa^3 t^2 (X \lrcorner \psi_\phi) \phi^* + s(X \lrcorner \bar{T}) \phi^*. \end{aligned}$$

Consequently

$$0 = \bar{\nabla}_X^s \phi^* + \frac{s}{t} (X \lrcorner \bar{\psi}_\phi) \phi^*.$$

Example 7.6. Let $(M^7, g, \xi, \eta, \psi)$ be an Einstein-Sasaki manifold with Killing vector ξ , Killing 1-form η and almost complex structure ψ on ξ^\perp . The Tanno deformation ($t > 0$)

$$g_t := tg + (t^2 - t)\eta \otimes \eta, \quad \xi_t := \frac{1}{t}\xi, \quad \eta_t := t\eta$$

has the property that $(M^7, g_t, \xi_t, \eta_t, \psi)$ remains Sasaki for all values of t . Call ∇^{g_t} the Levi-Civita connection of $(M, g_t, \xi_t, \eta_t, \psi)$ and T^{g_t} the characteristic torsion of the almost contact structure (a

characteristic connection exists since the manifold is Sasaki). Becker-Bender proved [Be12, Thm. 2.22] the existence of a Killing spinor with torsion for

$$\nabla_X^{g_t} + (\tfrac{1}{2t} - \tfrac{1}{2})(X \lrcorner T^{g_t}).$$

Quasi Killing spinors [FK00] are special instances of Definition 7.2 and produce gKST on the deformed Sasaki manifold $(M^7, g_t, \xi_t, \eta_t, \psi)$. As proved in [Be12], in this example generalised Killing spinors with torsion and Killing spinors with torsion are the same. Since the A of a gKS is symmetric, the G_2 -structure given by this spinor is cocalibrated (\mathcal{W}_{13}) .

Example 7.7. In [ABBK13] it was proved that on a nearly Kähler manifold the sets of ∇^c -parallel spinors, Riemannian Killing spinors, and Killing spinors with torsion coincide.

To conclude, the different existing notions of (generalised) Killing spinors (with torsion) are far from being disjoint and are best described, at least in dimensions 6 and 7, using the characterising spinor of the underlying G -structure as presented in this article.

Remark 7.8. At last note that the sign of the Killing constant may be reversed by choosing $j(\phi^*)$ instead of ϕ^* .

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